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CONTINUOUS FEEDBACK CONTROL OF PERTURBED MECHANICAL SYSTEMS†

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The problem of synthesizing continuous controls for a Lagrangian scleronomic mechanical system is investigated, on the assumption that the system is subject to uncontrollable bounded perturbations and that the vector of control forces is bounded in norm. a feedback control law is assumed, making it possible to steer the system to a given rest state in a finite time. The approach employed is based on methods of the theory of stability of motion. An implicitly given Lyapunov function is used to construct the control law and justify the construction. The existence of such a function is proved and its properties established. Results of a numerical simulation of the dynamics of various mechanical systems controllable in accordance with the proposed law are presented. It is shown that, for a point mass moving along a horizontal straight line, the control algorithm qualitatively approximates to time-optimal control. © 2003 Elsevier Science Ltd. All rights reserved.

There are two main groups of algorithms intended for controlling dynamical systems: algorithms that achieve (asymptotic) stability (stabilizing controls), and control laws that enable the system to be steered to a terminal state in a finite time. While in practice it is not possible to bring a system to a given state in a precise manner, and the system is brought to some neighbourhood of the state, the solution of such problems is preferably sought, from the standpoint of rapid response, using algorithms of the second group. As the terminal neighbourhood is diminished in size, the time of motion of a system controlled using algorithms of the second group becomes bounded, whereas when stabilizing controls are used this time increases without limit.

Another distinctive feature of formulations of control problems is the presence of constraints imposed on the control variables. In practice, such constraints are, as a rule, present in the control of mechanical systems.

Various control algorithms have been proposed [1–7] that enable a Lagrangian scleronomic system to be steered to a given terminal state in a finite time, on the assumption that the control forces are bounded and the system is subject to the action of uncontrollable perturbations. The steering may be achieved through approaches based on decomposition methods and the use of programmed trajectories [1–4], as well as linear feedback with piecewise-constant coefficients [5–7]. By using these and some other methods one obtains control laws which are described, generally speaking, by discontinuous functions of time and are therefore difficult to use in practice.

In this paper, an approach to the construction of continuous feedback control laws is proposed, which may be used to steer a mechanical system to a terminal state in a finite time. The control law proposed may be interpreted as linear feedback with coefficients which depend on the phase variables. The coefficients increase to infinity as the trajectory approaches the terminal state, but the control forces remain bounded and satisfy the conditions imposed on them.

1. FORMULATION OF THE PROBLEM

Consider a scleronomic mechanical system whose dynamics are described by Lagrange equations of the second kind:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = u + S \tag{1.1}$$

where $q, \dot{q} \in R^n$ are the vectors of generalized coordinates and velocities, $T(q, \dot{q}) = \langle A(q)\dot{q}, \dot{q}\rangle/2$ is the kinetic energy of the system (where $\langle \cdot, \cdot \rangle$ denotes the scalar product), u is the vector of generalized control forces, and S is a vector of unknown generalized forces, which we shall call perturbations.

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It is assumed that the positive-definite symmetric matrix of the kinetic energy $A(q) \in C^1$ is known, its eigenvalues for any q lie in an interval $[m, M]$, $0 < m \leqslant M$, that is

$$mz^2 \le \langle A(q)z, z\rangle \le Mz^2, \quad \forall q, z \in R^n \tag{1.2}$$

and the partial derivatives of $A(q)$ are uniformly bounded in norm, that is

$$|\partial A(q)/\partial q_i| \le C, \quad C > 0, \quad i = 1, \dots, n \tag{1.3}$$

Throughout, $z^2 = \langle z, z\rangle$, and $|\cdot|$ denotes the Euclidean norm of a vector or matrix (by the norm of a matrix we mean the norm of the corresponding linear operator in Euclidean space).

The vector of unknown perturbations $S(t, q, \dot{q})$ may be any vector-valued function, including a discontinuous one, that satisfies some existence conditions for a solution of system (1.1), as well as the condition

$$|S(t, q, \dot{q})| \le S_0, \quad S_0 > 0 \tag{1.4}$$

The vector of control forces is also subject to the restriction

$$|u| \le U, \; U > 0 \tag{1.5}$$

The phase coordinates q and the velocities $\dot{q}$ are assumed to be measurable at every instant of time.

Problem. Construct a control $u(q, \dot{q})$ as a continuous vector-valued function of the phase variables q and $\dot{q}$ in the set $R^{2n}\backslash\{(\bar{q}, 0)\}$, which satisfies condition (1.5), and designate a domain of admissible initial states such that any trajectory of system (1.1) beginning in that domain will reach the given terminal state $(\bar{q}, 0)$ in a finite time, whatever the perturbations S satisfying condition (1.4).

2. THE CONTROL LAW

We note that the terminal state is a rest state of the unperturbed system (1.1). Without loss of generality, let us assume that $\bar{q} = 0$, that is, the terminal state coincides with the origin of the phase state. This may be achieved by suitable choice of the generalized coordinates.

Define the control as

$$u(q, \dot{q}) = -a(q, \dot{q})A(q)\dot{q} - b(q, \dot{q})q \tag{2.1}$$

where

$$a(q, \dot{q}) = \sqrt{\frac{b(q, \dot{q})}{M}}, \quad b(q, \dot{q}) = \frac{3U^2}{8V(q, \dot{q})} \tag{2.2}$$

$$V(q, \dot{q}) = T + \frac{1}{2}b(q, \dot{q})q^2 + \frac{1}{2}a(q, \dot{q})\langle A(q)\dot{q}, q\rangle, \quad q^2 + \dot{q}^2 > 0 \tag{2.3}$$

Relationships (2.2) and (2.3) define the functions $a(q, \dot{q})$, $b(q, \dot{q})$ and $V(q, \dot{q})$ implicitly.

Theorem 1. In the domain $R^{2n}\backslash\{(0, 0)\}$ continuously differentiable functions $a(q, \dot{q})$, $b(q, \dot{q})$ and $V(q, \dot{q})$ exist satisfying (2.2) and (2.3), and such that $V > 0$.

Proof. Substituting the expressions for the functions $a(q, \dot{q})$ and $b(q, \dot{q})$ into Eq. (2.3) and transforming, we obtain

$$16V^2(q, \dot{q}) = 16T(q, \dot{q})V(q, \dot{q}) + 3U^2q^2 + \frac{2\sqrt{6}}{\sqrt{M}}\langle A(q)\dot{q}, q\rangle V^{1/2}(q, \dot{q}) \tag{2.4}$$

We introduce the notation

$$\begin{gathered} x = V^{1/2}(q, \dot{q}), \quad A_1(q, \dot{q}) = \langle A(q)\dot{q}, q\rangle \\ \alpha(q, \dot{q}) = 4T^{1/2}(q, \dot{q}), \quad \beta(q, \dot{q}) = \frac{2\sqrt{6}U}{\sqrt{M}}A_1(q, \dot{q}), \quad \gamma(q, \dot{q}) = \sqrt{3}U|q| \end{gathered} \tag{2.5}$$

and rewrite Eq. (2.4) in the form

$$F(q, \dot{q}, x) \stackrel{\text{def}}{=} 16x^4 - \alpha^2(q, \dot{q})x^2 - \beta(q, \dot{q})x - \gamma^2(q, \dot{q}) = 0 \tag{2.6}$$

Let us consider this equality as an equation in x. We will show that for any q, $\dot{q}$, $q^2 + \dot{q}^2 > 0$, a unique positive root of Eq. (2.6) exists and its multiplicity is one.

By Cauchy's inequality and condition (1.2), we have

$$\beta^2(q, \dot{q}) = \frac{24U^2}{M}A_1(q, \dot{q})^2 \leq \frac{24U^2}{M}T(q, \dot{q})\langle A(q)q, q\rangle \leq 24U^2T(q, \dot{q})q^2 = \alpha^2(q, \dot{q})\gamma^2(q, \dot{q})$$

whence it follows that

$$|\beta| \leq \gamma\alpha \tag{2.7}$$

We also note that, by formulae (2.5) and condition (2.1), the identity $\alpha(q, \dot{q}) = \gamma(q, \dot{q}) = 0$ cannot hold in the domain $R^{2n}\backslash\{(0, 0)\}$.

Considering, for a while, the coefficients α, β and γ to be fixed, let us prove the following auxiliary proposition.

Lemma. Every equation of the form

$$f(x) \stackrel{\text{def}}{=} 16x^4 - \alpha^2x^2 - \beta x - \gamma^2 = 0 \tag{2.8}$$

whose coefficients are constant and satisfy the inequality $\alpha^2 + \gamma^2 > 0$ and are connected by relation (2.7), has a unique positive real root and its multiplicity is one.

Proof. We will first prove that if α, $\gamma > 0$, exactly two real roots exist, one positive and one negative, and the positive root has multiplicity one.

Since $f(0) = -\gamma^2 < 0$, and for values of x with large absolute value one has $f(x) > 0$, Eq. (2.8) must have a positive and a negative root. Let us verify that there are no other real roots. Suppose, on the contrary, that any one of the equations of the family (2.7), (2.8), where α, $\gamma > 0$, has more than two real roots, that is, three or four. We shall show that there must then be an equation having a multiple real root.

It is elementary [8] that if exactly three roots exist, then one of them is multiple.

Now suppose the coefficients α_1, β_1, γ_1 are such that $|\beta_1| \leqslant \alpha_1\gamma_1$ and Eq. (2.8) has four real roots. For sufficiently large γ_2, Eq. (2.8) with coefficients α_1, β_1 and γ_2 has exactly two real roots and $|\beta_1| \leqslant \alpha_1\gamma_2$. Consequently, as the coefficient γ varies from γ_1 to γ_2 the number of roots changes, and γ_3 exists at which a multiple root x_0 of Eq. (2.8) appears. Obviously, this γ_3 also satisfies condition (2.7).

Since x_0 is a multiple root of the equation $f(x) = 0$ with coefficients α_1, β_1 and γ_3, it is also a root of the equation $f'(x) = 0$ with coefficients α_1 and β_1, which is

$$f'(x) = 64x^3 - 2\alpha_1^2x - \beta_1 = 0 \tag{2.9}$$

Multiply Eq. (2.8), where $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \gamma_3$, by -4, and Eq. (2.9) by x, and add the resulting equations together. This gives the equation

$$2\alpha_1^2x^2 + 3\beta_1x + 4\gamma_3^2 = 0 \tag{2.10}$$

which must have x_0 as a root. By condition (2.7) and the inequalities α_1, $\gamma_3 > 0$, the discriminant $D = 9\beta_1^2 - 32\alpha_1^2\gamma_3^2$ of Eq. (2.10) is negative, and therefore Eq. (2.7) has no real roots. This contradiction proves the truth of the statement of the lemma in the case when α, $\gamma > 0$.

Now consider the case in which $\alpha = 0$ or $\gamma = 0$. If $\alpha = 0$, then by Eq. (2.7) $\beta = 0$, Eq. (2.8) becomes $x^4 = \gamma^2/16$, $\gamma > 0$, which has a single positive root and the multiplicity of the root is one.

In the case when $\gamma = 0$, Eq. (2.7) implies that $\beta = 0$, and Eq. (2.8) reduces to the equation $x^4 = \alpha^2x^2$, $\alpha > 0$, which has three real roots. One of these roots, viz. $x = 0$, has multiplicity two, and the single positive root is of multiplicity one.

We now continue the proof of Theorem 1, considering the coefficients α, β and γ as functions of the phase variables q and $\dot{q}$, It follows from the statement of the lemma that, for any $(q, \dot{q}) \in R^{2n}\backslash\{(0, 0)\}$, the polynomial equation (2.6) has a unique positive real root $x_0(q, \dot{q})$ and its multiplicity is one. Consequently

$$\partial F/\partial x(q, \dot{q}, x_0(q, \dot{q})) \neq 0, \quad (q, \dot{q}) \in R^{2n}\backslash\{(0, 0)\}$$

and by the Implicit Function Theorem $x_0(q, \dot{q}) \in C^1(R^{2n}\backslash\{(0, 0)\})$. Then the function $V(q, \dot{q}) = x_0^2(q, \dot{q})$, and together with it also the functions $a(q, \dot{q})$ and $b(q, \dot{q})$ defined by (2.2), are continuously differentiable in the domain $R^{2n}\backslash\{(0, 0)\}$, and moreover $V > 0$. The theorem is proved.

Bearing in mind the statement of Theorem 1 and formula (2.1), we conclude that the control function $u(q, \dot{q})$ is defined and continuously differentiable in the domain $R^{2n}\backslash\{(0, 0)\}$.

3. JUSTIFICATION OF THE CONTROL LAW

We shall now find the domain of admissible initial states and show that any trajectory of system (1.1), (2.1) beginning in that domain will reach the origin in a finite time. This will be done by methods of stability theory, and we will show that the function V is a Lyapunov function of the system under consideration.

Let us find upper and lower bounds for $V(q, \dot{q})$. Using Cauchy's inequality, formulae (2.2), conditions (1.2) and the notation (2.5), we obtain

$$|a(q, \dot{q})A_1(q, \dot{q})| \leq |a(q, \dot{q})|(2T(q, \dot{q})\langle A(q)q, q\rangle)^{1/2} \leq$$
$$\leq T(q, \dot{q}) + \frac{1}{2}a^2(q, \dot{q})\langle A(q)q, q\rangle \leq T(q, \dot{q}) + \frac{1}{2}b(q, \dot{q})q^2$$

and this implies the limits

$$V_-(q, \dot{q}) \leq V(q, \dot{q}) \leq 3V_-(q, \dot{q}) \tag{3.1}$$

where

$$V_-(q, \dot{q}) = \frac{1}{4}(2T(q, \dot{q}) + b(q, \dot{q})q^2) \tag{3.2}$$

Let us substitute expressions (2.2) and (3.2) for the functions $b(q, \dot{q})$ and $V_-(q, \dot{q})$ into estimates (3.1). After some reduction, we obtain the inequalities

$$\xi(q, \dot{q}) \leq 32V^2(q, \dot{q}) \leq 3\xi(q, \dot{q}), \quad \xi(q, \dot{q}) = 16T(q, \dot{q})V(q, \dot{q}) + 3U^2q^2$$

whence, solving for $V(q, \dot{q})$, we arrive at the following limits (for brevity, we shall henceforth omit the arguments and q and $\dot{q}$)

$$\frac{1}{4}T(1 + \sqrt{1 + 3\lambda}) \leq V \leq \frac{3}{4}T(1 + \sqrt{1 + \lambda}), \quad \lambda = \frac{U^2q^2}{2T^2} \tag{3.3}$$

Note that the functions of the phase variables q and $\dot{q}$ on the right and left of (3.3) are expressed explicitly in terms of the known parameters of the problem. It follows from the form of (3.3) that the function $V(q, \dot{q})$ can be defined as zero at (0, 0) while still remaining continuous, but it will not be differentiable there. The form of the function V for a mechanical system consisting of a point mass moving along a horizontal line will be presented in Section 5.

We will now evaluate the derivative $\dot{V}$. Differentiating the functions $a(q, \dot{q})$, $b(q, \dot{q})$ and $V(q, \dot{q})$ along trajectories of system (1.1), (2.1), we obtain

$$\begin{gathered} \dot{a} = -\frac{a}{2V}\dot{V}, \quad \dot{b} = -\frac{b}{V}\dot{V} \\ \dot{V} = \dot{T} + b\langle q, \dot{q}\rangle + aT + \frac{a}{2}\left\langle \frac{d}{dt}A\dot{q}, q\right\rangle - \frac{\dot{V}}{2V}\left(bq^2 + \frac{a}{2}A_1\right) \end{gathered} \tag{3.4}$$

By the theorem on the variation of the kinetic energy of a scleronomic Lagrangian system, and by the definition of the vector-valued function $u(q, \dot{q})$, we have

$$\dot{T} = \langle u + S, \dot{q}\rangle = -2aT - b\langle \dot{q}, q\rangle + \langle S, \dot{q}\rangle \tag{3.5}$$

and by Eqs (1.1)

$$\frac{d}{dt}A\dot{q} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial q} + u + S \tag{3.6}$$

Substituting Eqs (3.5), (3.6) and (2.1) into the last equation of (3.4), we obtain the following relation for the derivative of the function $V(q,\dot q)$ (throughout what follows, summation is from $i = 1$ to $i = n$)

$$\dot V = -a\left(T+\frac{a}{2}A_1+\frac{b}{2}q^2\right)+\frac{a}{4}\left\langle\left(\sum\frac{\partial A}{\partial q_i}q_i\right)\dot q,\dot q\right\rangle-\frac{\dot V}{2V}\left(bq^2+\frac{a}{2}A_1\right)+\left\langle S,\dot q+\frac{a}{2}q\right\rangle \tag{3.7}$$

Let us transform and estimate the separate terms on the right of Eq. (3.7). It follows from the definitions (2.2), (2.3) and (2.5) of the functions $a(q,\dot q)$, $V(q,\dot q)$ and $A_1(q,\dot q)$ that

$$a\left(T+\frac{a}{2}A_1+\frac{b}{2}q^2\right) = aV = \frac{\sqrt{3}U}{2\sqrt{2M}}V^{1/2} \tag{3.8}$$

By the Cauchy inequality, condition (1.2) and relations (2.2), (3.1) and (3.2), we have

$$\left|\dot q+\frac{a}{2}q\right|^2 = \dot q^2+\frac{a^2}{4}q^2+a\langle\dot q,q\rangle\le\frac{5}{4}(\dot q^2+a^2q^2)\le\frac{5}{4}\left(\frac{2}{m}T+\frac{b}{M}q^2\right)\le\frac{5}{m}V_-\le\frac{5}{m}V$$

Hence, by condition (1.4), we have the inequality

$$\left|\left\langle S,\dot q+\frac{a}{2}q\right\rangle\right|\le S_0\sqrt{\frac{5}{m}}V^{1/2} \tag{3.9}$$

Using condition (1.3) and the estimate

$$\sum|q_i|\le\sqrt{n}|q|$$

we can conclude that

$$\left|\sum\frac{\partial A}{\partial q_i}q_i\right|\le\sqrt{n}C|q| \tag{3.10}$$

Relations (1.2), (2.2), (3.1) and (3.2) imply the following inequalities

$$q^2\le\frac{4}{b}V_-\le\frac{4}{Ma^2}V,\quad \dot q^2\le\frac{2}{m}T\le\frac{4}{m}V_-\le\frac{4}{m}V$$

Hence, by inequality (3.10), we obtain

$$\left|\frac{a}{4}\left\langle\left(\sum\frac{\partial A}{\partial q_i}q_i\right)\dot q,\dot q\right\rangle\right|\le\frac{a}{4}\sqrt{n}C|q|\dot q^2\le\frac{2\sqrt{n}C}{m\sqrt{M}}V^{3/2} \tag{3.11}$$

Substituting relations (3.8), (3.9) and (3.11) into expression (3.7) for the derivative $\dot V(q,\dot q)$ and transferring the last term in (3.7) to the left, we arrive at the inequality

$$B\dot V\le-\delta(q,\dot q)V^{1/2} \tag{3.12}$$

where

$$\delta(q,\dot q) = \frac{\sqrt{3}U}{2\sqrt{2M}}-S_0\sqrt{\frac{5}{m}}-\frac{2\sqrt{n}C}{m\sqrt{M}}V(q,\dot q)$$

$$B(q,\dot q) = 1+\frac{b(q,\dot q)}{2V(q,\dot q)}q^2+\frac{a(q,\dot q)}{4V(q,\dot q)}A_1(q,\dot q) = \frac{1}{V}\left(T+bq^2+\frac{3a}{4}A_1\right)$$

(the last equality in the chain is obtained using (2.3)).

By the Cauchy inequality, formula (2.2) and condition (1.2)

$$\left|\frac{3}{4}aA_1(q,\dot q)\right|\le\frac{1}{2}T(q,\dot q)+\frac{9a^2}{16}\langle Aq,q\rangle\le\frac{1}{2}T+\frac{3b}{4}q^2$$

$$\left|\frac{3}{4}aA_1(q,\dot q)\right|\le\frac{3}{2}T(q,\dot q)+\frac{3a^2}{16}\langle Aq,q\rangle\le 2T+\frac{b}{2}q^2$$

Using the first inequality to estimate the function $B(q,\dot q)$ from below and the second to estimate it from above, we obtain

$$0 < \frac{1}{4V}(2T + bq^2) \le B(q,\dot q) \le \frac{3}{V}\left(T + \frac{b}{2}q^2 + \frac{a}{2}A_1\right) = 3 \tag{3.13}$$

Since $B(q, \dot q) > 0$, it follows that the sufficient condition for the derivative $\dot V(q, \dot q)$ to be negative is that the expression in parentheses on the right of inequality (3.12) should be negative. Putting

$$V(t) = V(q(t), \dot q(t)),\ B(t) = B(q(t), \dot q(t)),\ \delta(t) = \delta(q(t), \dot q(t)) \tag{3.14}$$

we write inequality (3.12) in the form

$$\dot V(t) \le -\frac{\delta(t)}{B(t)} V^{1/2}(t) \tag{3.15}$$

Theorem 2. Suppose the following condition is satisfied at the initial time t_0

$$\delta(t_0) > 0 \tag{3.16}$$

Then the derivative of the function V along trajectories of system (1.1), (2.1) will satisfy the inequality

$$\dot V(t) \le -\frac{\delta(t_0)}{3} V^{1/2}(t),\ t \ge t_0 \tag{3.17}$$

Proof. It follows from relations (3.13), (3.15) and from condition (3.16) that

$$\dot V(t_0) \le -\frac{\delta(t_0)}{B(t_0)} V^{1/2}(t_0) \le -\frac{\delta(t_0)}{3} V^{1/2}(t_0) < 0$$

Consequently, for $t > t_0$ in a sufficiently small neighbourhood of the point t_0, we have the inequality $V(t) < V(t_0)$. This inequality turns out to be true for all $t > t_0$.

Indeed, suppose the contrary. Let $t' > t_0$ be the first time at which V again takes the value $V(t_0)$. Then $V(t) < V(t_0)$ for $t \in (t_0, t')$. Hence we conclude from the form (3.14) of the function $\delta(t)$ and from condition (3.16) that

$$\delta(t) > \delta(t_0) > 0 \tag{3.18}$$

and so, by (3.15), we obtain the inequality $\dot V(t) < 0$ for $t \in (t_0, t')$.

On the other hand, since $V(t_0) = V(t')$, it follows by Lagrange's Theorem that a point $t'' \in (t_0, t')$ exists such that $\dot V(t'') = 0$. This contradiction shows that $V(t) < V(t_0)$ for $t > t_0$.

The inequality just proved implies the validity of the estimate $\delta(t) > \delta(t_0) > 0$ for all $t > t_0$, whence, by relations (3.13) and (3.15), we obtain inequality (3.17), which proves the theorem.

It follows from relations (3.3) and (3.17) that the value of the function V along a trajectory of system (1.1), (2.1) will tend to zero, while the trajectory itself will approach the origin.

To estimate the time of motion, let us integrate inequality (3.17) over the interval $[t_0, t]$. We obtain

$$t - t_0 \le \frac{6}{\delta(t_0)}(V^{1/2}(t_0) - V^{1/2}(t))$$

Taking into account that $V(t) \to 0$ as t increases, we obtain the following estimate for the time τ taken by system (1.1), (2.1) to move from the origin $q_0 = q(t_0)$, $\dot q_0 = \dot q(t_0)$ to the terminal state $q = \dot q = 0$

$$\tau \le 6V^{1/2}(q_0, \dot q_0)/\delta(q_0, \dot q_0) \tag{3.19}$$

We will now verify that the vector-valued control function $u(q, \dot q)$ satisfies condition (1.5). By the Cauchy inequality we have

$$u^2 = a^2|A\dot q|^2 + b^2q^2 + 2abA_1 \le \frac{4}{3}(a^2|A\dot q|^2 + b^2q^2 + abA_1) \tag{3.20}$$

Since $A(q)$ is a positive-definite symmetric matrix satisfying conditions (1.2), the matrix $A^{-1}(q)$ is also positive-definite and symmetric, and its eigenvalues lie in the interval $[1/M, 1/m]$ (see, e.g. [8]). Consequently

$$z^2/M \leq \langle A^{-1}(q)z, z\rangle, \quad \forall q, \ z \in R^n$$

Substituting $z = A(q)\dot{q}$ into this inequality, we obtain the relations

$$|A\dot{q}|^2 = z^2 \leq M\langle A^{-1}z, z\rangle = 2MT$$

using which we can extend the limit (3.20) as follows:

$$u^2 \leq \frac{4}{3}(2Ma^2T + b^2q^2 + abA_1)$$

Using expressions (2.2) and (2.3) for the functions a and V, we obtain the inequality

$$u^2 \leq \frac{8b}{3}V = U^2$$

from which it follows that constraints (1.5) hold along the trajectory of system (1.1), (2.1).

Some properties of the vector-valued control function $u(q, \dot{q})$ will now be noted. We will calculate its values in the subspaces $q = 0$ and $\dot{q} = 0$ of the phase space $(q, \dot{q}) \in R^{2n}$. Let $q = 0$. Then

$$V(0, \dot{q}) = T(0, \dot{q}), \quad a(0, \dot{q}) = \frac{\sqrt{3}U}{2(2MT(0, \dot{q}))^{1/2}}$$

Consequently

$$u(0, \dot{q}) = -\frac{\sqrt{3}U}{2(2MT(0, \dot{q}))^{1/2}}A(0)\dot{q} = -\frac{\sqrt{3}U}{2(2MT(0, \dot{e}))^{1/2}}A(0)\dot{e} \tag{3.21}$$

where e is a unit vector collinear with $\dot{q}$.

If $\dot{q} = 0$, then

$$V(q, 0) = \frac{b(q, 0)}{2}q^2, \quad b(q, 0) = \frac{3U^2}{4b(q, 0)q^2}$$

whence we get

$$b(q, 0) = \frac{\sqrt{3}U}{2|q|}, \quad u(q, 0) = -\frac{\sqrt{3}U}{2|q|}q \tag{3.22}$$

Thus, the control force vector $u(q, \dot{q})$ is constant in the subspaces $q = 0$ and $\dot{q} = 0$ along any straight line passing through the origin of the phase space, and it points toward the origin.

The proposed control law may be formulated without using the functions $a(q, \dot{q})$ and $b(q, \dot{q})$. To that end, we transform expression (2.1) for $u(q, \dot{q})$ by substituting into it formulae (2.2) for a and b. This gives a new definition of the vector-valued control function

$$u(q, \dot{q}) = -\frac{\sqrt{3}U}{2(2MV(q, \dot{q}))^{1/2}}\dot{q} - \frac{3U^2}{8V(q, \dot{q})}q$$

where the function $V(q, \dot{q})$ is implicitly defined by Eq. (2.4).

4. THE SUFFICIENT CONDITIONS FOR EFFECTIVE CONTROL

Formulae (3.14) and (3.16) imply the following sufficient conditions for the system to reach the given terminal state

$$U > S_* + \frac{4\sqrt{2nC}}{\sqrt{3m}}V(q_0, \dot{q}_0), \ S_* = 2\sqrt{\frac{10M}{3m}}S_0 \tag{4.1}$$

This condition relates the maximum admissible value of the control U and of the perturbations S_0 with the domain of admissible initial states of the system. In particular, in a neighbourhood of the terminal state where the function $V(q, \dot{q})$ is small, condition (4.1) may be written in the form

$$U > S_*$$

This condition characterizes the excess of the control forces over the perturbations which is sufficient for the control objective to be achieved.

If there are no perturbations, i.e. $S_0 = 0$, the proposed control law will steer system (1.1) to the terminal state in a finite time from any point of the domain of admissible initial states, which is given by the inequality

$$V(q, \dot{q}) \le \sqrt{\frac{3}{2n}}\frac{mU}{4C}$$

Taking note of relation (3.3), we can state that this domain will certainly contain the ellipsoid

$$T(q, \dot{q}) + \left(T^2(q, \dot{q}) + \frac{U^2}{2}q^2\right)^{1/2} \le \frac{mU}{\sqrt{6nC}}$$

Note that the control law defined by relations (2.1)–(2.3) is independent of the constants S_0 and C and of the initial state $(q_0, \dot{q}_0)$. It may therefore be formally applied even if inequality (4.1) fails to hold. Computer simulation of the dynamics of various systems shows that the control law is effective far beyond the limits of the sufficient conditions (4.1). This is due to the fact that condition (4.1) guarantees a monotone decrease of the function V along a trajectory of system (1.1), when the control is by the law (2.1)–(2.3). However, the function V may tend to zero in a non-monotone manner, and the trajectories of the system will then arrive at the terminal state as before. The simulation results presented below will illustrate this behaviour of the system.

5. RESULTS OF A SIMULATION

To verify the effectiveness of the proposed control law and to illustrate its operation, numerical simulation was carried out for controlled motions of a two-link system on a stationary base. It was assumed that the system is moving in a horizontal plane, that is, the gravity force was not taken into account. The generalized coordinates q_1 and q_2 of the system were taken to be the joint angles of the links in a fixed system of coordinates (see the inset to Fig. 1).

The kinetic energy matrix of the system has the form

$$A(q) = \begin{Vmatrix} R_1 & R_3\cos(q_1 - q_2) \\ R_3\cos(q_1 - q_2) & R_2 \end{Vmatrix}$$

and the equations of motion may be written as

$$\begin{aligned} \ddot{q}_1 &= (R_2Q_1 - R_3Q_2\cos(q_1 - q_2))/\Delta \\ \ddot{q}_2 &= (R_1Q_2 - R_3Q_1\cos(q_1 - q_2))/\Delta \end{aligned} \tag{5.1}$$

where

$$Q_1 = u_1 + S_1 - R_3\sin(q_1 - q_2)\dot{q}_2^2, \ Q_2 = u_2 + S_2 + R_3\sin(q_1 - q_2)\dot{q}_1^2$$

$$\Delta = \det A = R_1R_2 - R_3^2\cos^2(q_1 - q_2)$$

The computations were carried out for the following parameter values: $R_1 = 13.9$, $R_2 = 2.1$ and $R_3 = 3$ kg·m². Under these conditions the eigenvalues of the inertia matrix turned out to be between

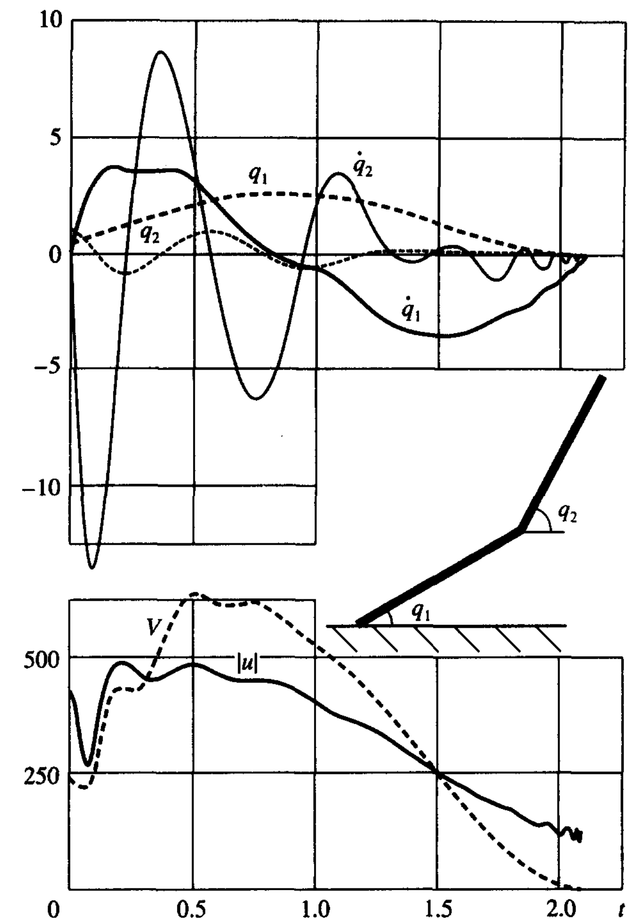

Fig. 1

the constants $m = 1.4$ and $M = 14.6$ kg·m^2, and its partial derivatives were bounded in norm by the constant $C = 3$. The maximum admissible norm of the vector of control moments was taken to be $U = 15$ N·m.

The two-link system was brought from the initial state $q_1 = 0.5, q_2 = 1$ rad, $\dot{q}_1 = \dot{q}_2 = 0$ rad/sec (this is the state illustrated in the inset to Fig. 1) to the "horizontal stretched arm" position $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$.

The system of equations (5.1) was integrated by the Runge–Kutta method. Integration was broken off when the quantity $(q_1^2 + q_2^2 + \dot{q}_1^2 + \dot{q}_2^2)^{1/2}$ – the Euclidean distance between the actual state of the system and the origin in the phase space – fell below 0.005.

With the parameters thus chosen, the sufficient condition (4.1) for steering the mechanical system to the terminal state using the proposed control law may be rewritten as

$$U > 11.8S_0 + 9.9V(q_0, \dot{q}_0)$$

In the simulation, the perturbing moment was defined by a constant vector-valued function $S(t) = (0, 250)$. Consequently, the magnitude of the perturbation vector in constraint (1.4) will not exceed the quantity $S_0 = 250$ in norm, and the sufficient condition for the system to be steered to the terminal state becomes

$$U > 2950 + 9.9V(q_0, \dot{q}_0)$$

that is, for the selected value of U, the condition will not be satisfied for any initial values of the phase variables. Nevertheless, the proposed control law will overcome the perturbations and steer the system to the terminal state.

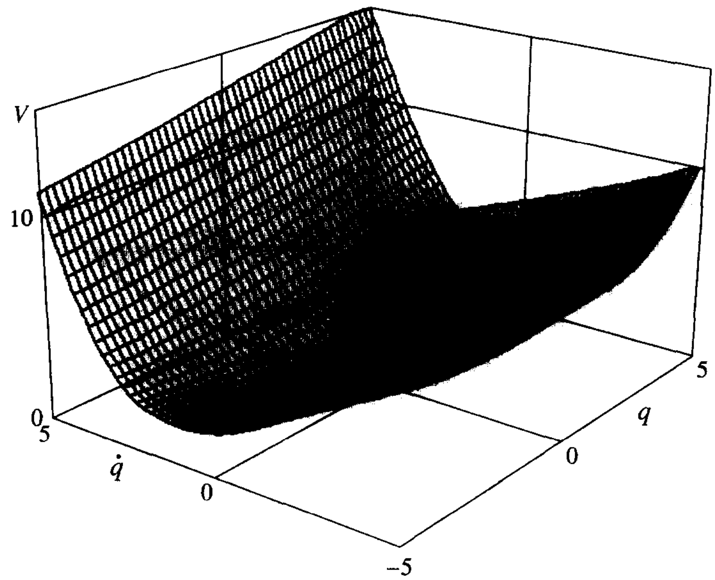

Fig. 2

The phase variables of the system are plotted in the upper part of Fig. 1 as functions of the phase variables. The dashed curves correspond to the generalized coordinates (rad), the solid curves correspond to the generalized velocities (rad/sec), the thickened curves describe the motion of the first link and the thin curves describe that of the second.

The magnitude of the vector of control moments $|u|$ is plotted in the lower part of Fig. 1 as a function of time, together with the behaviour of the function V along the trajectory under consideration. Clearly, V tends to zero and the control satisfies the restriction $|u| < 500$. As already remarked, V tends to zero non-monotonically, because the sufficient condition (4.1) for the derivative $\dot{V}$ to be negative is not satisfied.

In accordance with the algorithm, the control u is defined in terms of the function V, which is defined implicitly by Eq. (2.4). The quantity $x = \sqrt{V}$, as a root of the fourth-order polynomial equation (2.6), and hence also the function V, may be expressed analytically using Cardano's formulae. However, there is no need for an explicit representation of the function V when running the algorithm. From a computational point of view, it is more convenient to find the present value of the function by solving Eq. (2.6) numerically, say by Newton's method. At each step of the integration, it is convenient to take the value of V from the previous step as the initial approximation. Since the function V decreases monotonically along the trajectory and is continuous, the old value is slightly larger than the new (unknown) one.

Let us investigate the limiting possibilities of the algorithm by numerical simulation and compare it with a time-optimal control law. To that end, we consider the mechanical system consisting of a point mass of unit mass moving along a horizontal straight line. The equations of motion of the point are

$$\ddot{q} = u + S \tag{5.2}$$

where q is the coordinate of the point on the line. The control force u and the perturbations S are subject to the restrictions

$$|u| \leq 1, \ \ |S| \leq S_0 < 1 \tag{5.3}$$

Due to the simplicity and low dimensionality of the system, one can express the functions $V(q, \dot{q})$, $u(q, \dot{q})$ and the other characteristics of the motion by graphical means. Figure 2 is a graph of the function $V(q, \dot{q})$, in terms of which the control law (2.1), (2.2) is expressed and which is a Lyapunov function for system (5.2).

Figure 3 is a phase portrait of system (5.2) for the case in which there are no perturbations, that is, $S \equiv 0$. The solid curves are the phase trajectories of the motion of the point mass, and the dashed curves are level curves of the function V.

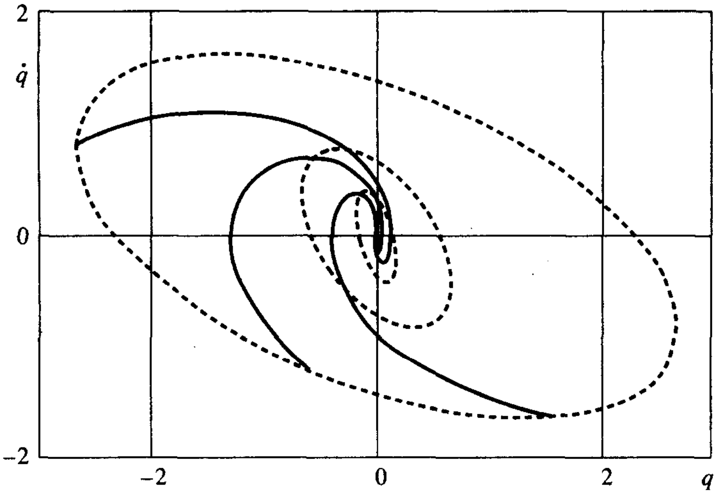

Fig. 3

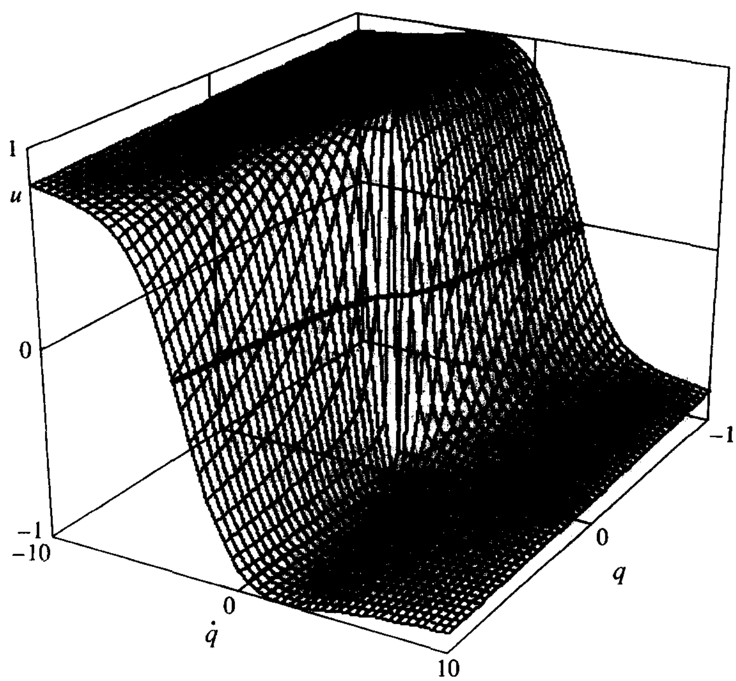

Fig. 4

By the definition of V, it is symmetrical about the origin, that is, $V(q, \dot{q}) = V(-q, -\dot{q})$. The functions a, b and u have the same property. Hence the phase portrait of the unperturbed system is also symmetrical about the point (0, 0).

A graph of the control function $u(q, \dot{q})$ is shown in Fig. 4. It follows from formulae (3.21) and (3.22) that, for the parameter values chosen, the function u satisfies the following relations on the straight lines $q = 0$ and $\dot{q} = 0$

$$u(0, \dot{q}) = u(q, 0) = \sqrt{3}/2$$

It can be seen that in the domains $q, \dot{q} > 0$ and $q, \dot{q} < 0$, and at values of the velocity $\dot{q}$ of large magnitude, the surface shown in Fig. 4 has almost horizontal parts corresponding to values of u close to $\pm\sqrt{3}/2$.

As might have been expected, in the neighbourhood of the origin the function $u(q, \dot{q})$ has partial derivatives of arbitrarily large magnitude. This is because the control force must cope with any perturbations, including discontinuous ones, that satisfy condition (5.3) and guarantee a monotone decrease of the function V along the trajectory. Consequently, the closer to the origin, the higher must the admissible rate of variation of the control force u be (at the point (0, 0) itself the function u is not defined).

Let us find the curve on which the control changes sign. It is clear from Fig. 4 that this curve is defined by the equation $u(q, \dot{q}) = 0$. Hence, by (2.1), taking into account that system (5.2) satisfies the equalities $A(q) = m = M = 1$ (the notation being the same as in Section 1), we obtain

$$a(q, \dot{q}) = -\dot{q}/q \tag{5.4}$$

Using relations (2.2), let us express the functions $V(q, \dot{q})$ and $b(q, \dot{q})$ in terms of $a(q, \dot{q})$ and substitute the result into the above expression (5.4). Taking into account that $U = 1$ and transforming, we obtain the equation $3q^2 = 4\dot{q}^4$. The function $a(q, \dot{q})$ is positive by definition. Hence, by Eq. (5.4), the coordinate q and velocity $\dot{q}$ have different signs at each point of the required curve. Therefore, the curve itself is defined by

$$q = \begin{cases} -2\sqrt{3}\dot{q}^2/3, & \dot{q} > 0 \\ 2\sqrt{3}\dot{q}^2/3, & \dot{q} < 0 \end{cases} \tag{5.5}$$

and it consists of two branches of parabolas which are symmetrical about the origin (the solid curve in Fig. 4).

Let us compare the control law described above with a control that minimizes the time of the motion. It is well known [9] that for system (5.2) with $S \equiv 0$ the time-optimal control has the form

$$u_{\text{opt}} = \begin{cases} -1, & \text{if} \quad \dot{q} > 0 \quad \text{and} \quad q \geq -\dot{q}^2/2 \\ -1, & \text{if} \quad \dot{q} < 0 \quad \text{and} \quad q > \dot{q}^2/2 \\ 1 & \text{otherwise} \end{cases} \tag{5.6}$$

The phase space $(q, \dot{q})$ is divided into two domains by a curve, called the switching curve (SC), consisting of two branches of parabolas $q = \pm\dot{q}^2/2$ which are symmetrical about the origin. The function u_{opt} takes two values: above the SC and on its left branch $u_{\text{opt}} = -1$, and below the SC and on its right branch $u_{\text{opt}} = 1$. The function u, whose graph is shown in Fig. 3, and the function u_{opt} given by (5.6) are readily seen to be qualitatively similar. The SC for u_{opt} and its analogue (5.5) for the control u are each the union of two branches of parabolas in which the coefficients of $\dot{q}^2$ are $1/2$ and $2\sqrt{3}/3$, respectively.

It perturbations appear in system (5.2), that is, the condition $S \equiv 0$ is not assumed, then the time-optimal control law becomes [10]

$$u'_{\text{opt}} = \begin{cases} -1, & \text{if} \quad \dot{q} > 0 \quad \text{and} \quad q \geq -\dot{q}^2/(2(1-S_0)) \\ -1, & \text{if} \quad \dot{q} < 0 \quad \text{and} \quad q > \dot{q}^2/(2(1-S_0)) \\ 1 & \text{otherwise} \end{cases} \tag{5.7}$$

In that case the SC is the union of two branches of parabolas $q = \pm\dot{q}^2/(2(1-S_0))$, coinciding with the curve $u(q, \dot{q})$ given by formulae (5.5) with $S_0 = 1 - \sqrt{3}/4$.

Figure 5 shows a graph of the function $r(q, \dot{q})$ for system (5.2), under the control prescribed by the law (2.1)–(2.3) when there are no perturbations, that is, when $S \equiv 0$. By definition, the value of this function at each point of the phase space equals the time it takes for the system to move from that point to the terminal state. For comparison, Fig. 6 is a graph of the Bellman function, equal at each point to the minimum possible time of motion from that point to the terminal state. It can be seen that the time of motion of the system controlled using the proposed algorithm is approximately 1.5 times the minimum time.

In order to determine the limiting possibilities of our control algorithm, the motion of system (5.2) was simulated numerically for the case of perturbations defined by

$$S(q, \dot{q}) = -S_0 u_{\text{opt}}$$

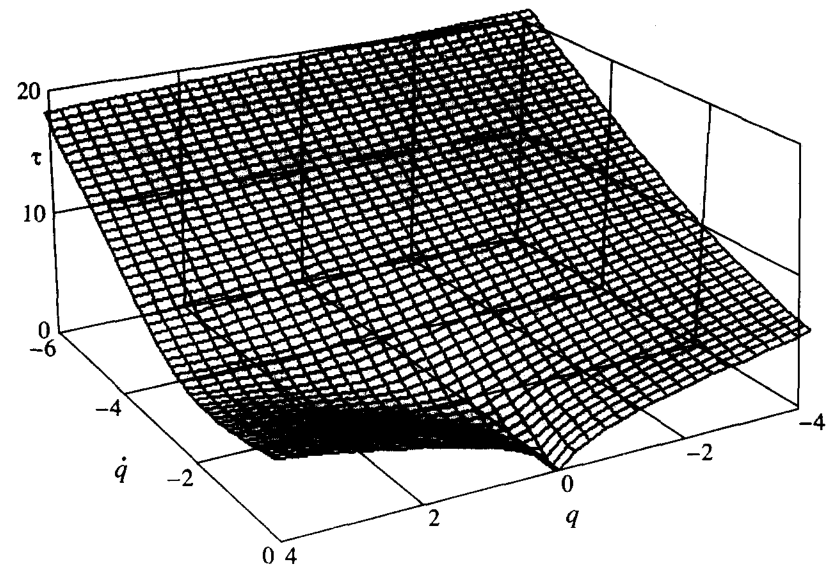

Fig. 5

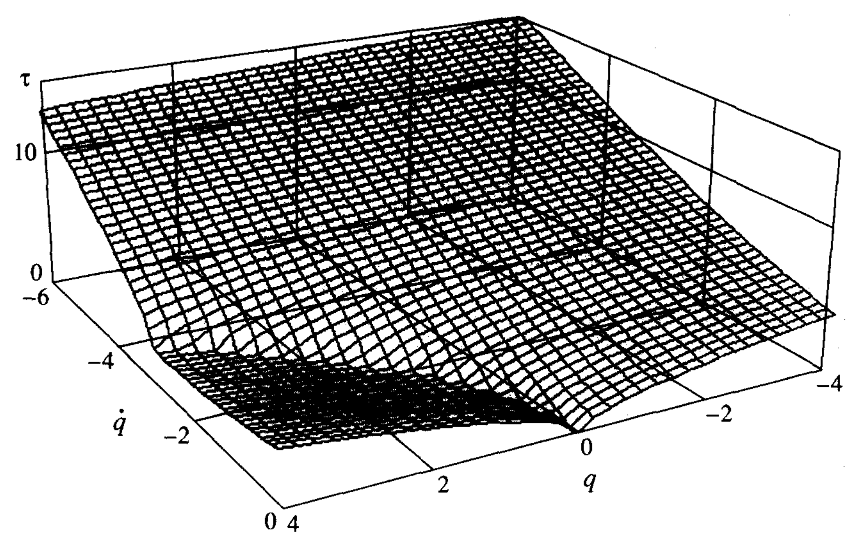

Fig. 6

where u_{opt} is defined by formula (5.6); this was done for different S_0 values. It turned out that the limiting value of the perturbations S_0 for which the system is brought to the terminal state by algorithm (2.1)–(2.3) is approximately $0.87 \approx \sqrt{3}/2$, that is, the value of the function u on the straight line $q = 0$. This is the value corresponding to the horizontal sections of the graph described above. We recall [10] that, for the optimal control law, the system may be brought to the terminal time if and only if $S_0 < 1$.

Thus, the approach adopted above enables us to construct algorithms that yield bounded controls which are smooth functions of the phase variables and time. These algorithms can be used to control any scleronomic mechanical system and enable it to be steered to a given terminal state in a finite time. The results of the numerical simulation of controlled motions of a point mass along a horizontal straight line demonstrate that the proposed control law is qualitatively similar to a time-optimal control law.

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